Muon Tech Note
MU - 026

Stability of Alternating Solenoid Lattice Solutions

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11 January 1999

We examine the stability of particle orbits in alternating solenoid lattices. We show that the equations of motion reduce to a form of the Mathieu equation. Stable orbits are determined by particular values of the parameters of the Mathieu equation. These parameters are then related to the momentum of the particle and the field strength and period of the alternating solenoid lattice. Stability predictions are shown to agree exactly for a perfect sinusoidally-varying magnetic field. The stability predictions do not work for the alternating solenoid magnetic field pattern used in the muon collider, except for the fundamental resonance condition.

1 Introduction

An alternating solenoid lattice has been designed by Palmer for transverse emittance cooling of muon beams at a muon collider [1-3]. The intention is to scale this lattice for use over a range of incident transverse emittances that vary by a factor of about 300. Thus the question arises of how to properly transform the lattice parameters in such a way as to obtain efficient cooling, while at the same time avoiding unstable regimes that are known to exist in alternating solenoid lattices [4].

2 Relation to the Mathieu equation

The radial equation of motion of a charged particle in an electromagnetic field is given in cylindrical coordinates as

$$\ddot{r} - r\dot{\phi}^2 = \frac{e}{m\gamma}\left(\mathbf{E}_r - r\dot{\phi}B_Z\right) \tag{1}$$

For the case considered here E_r is 0. The particle develops an angular momentum from crossing the fringe field at the end of the solenoid given by

$$p_\phi = -\frac{eB_Z r}{2} = m\gamma v_\phi = m\gamma r\dot{\phi} \tag{2}$$

We can use this relation to obtain the azimuthal velocity

$$\dot{\phi} = -\frac{eB_Z}{2m\gamma} \tag{3}$$

Note that this is also the solution of the ϕ equation of motion in cylindrical coordinates when there is no azimuthal acceleration. Substituting Eq. 3 back into Eq. 1, we find

$$\ddot{r} - \frac{r}{4}\left(\frac{eB_Z}{m\gamma}\right)^2 = 0 \tag{4}$$

Now approximate the field pattern in an alternating solenoid lattice with the leading term of a Fourier expansion

$$B_Z = B_o \sin kz \tag{5}$$

Substituting Eq. 5 back into Eq. 4 gives

$$\ddot{r} - r\left(\frac{e\,B_o}{2\,m\,\gamma}\right)^2 \sin^2 kz = 0 \tag{6}$$

Using the relations

$$\begin{aligned} z &= v\,t \\ \sin^2 x &= \tfrac{1}{2}\,(1 - \cos 2x) \end{aligned} \tag{7}$$

we can transform Eq. 6 into the form

$$\frac{d^2 r}{dz^2} - [A - A\cos 2kz]\,r = 0 \tag{8}$$

where

$$A = \tfrac{1}{2}\left(\frac{e\,B_o}{2\,m\,v\,\gamma}\right)^2 \tag{9}$$

Finally, the transformation

$$u = k\,z \tag{10}$$

allows us to write Eq. 8 as [4,6]

$$\frac{d^2 r}{du^2} - \left[\frac{A}{k^2} - \frac{A}{k^2}\cos 2u\right] r = 0 \tag{11}$$

This can be compared with the canonical form of the Mathieu equation [7]

$$\frac{d^2 y}{d v^2} - [a - 2\,q\,\cos 2v]\,y = 0 \tag{12}$$

Thus we can identify the Mathieu parameters as

$$\begin{aligned} q &= \left(\frac{e\,B_o}{4\,m\,v\,\gamma\,k}\right)^2 \\ a &= 2\,q \end{aligned} \tag{13}$$

If we use the relation

$$k = \frac{2\pi}{\lambda} \quad \textbf{(14)}$$

where λ is the spatial period of the magnetic field, we can specify the Mathieu parameter q in terms of the momentum of the particle and the magnetic field parameters as

$$q = \left(\frac{e B_o \lambda}{8 \pi p} \right)^2 \quad \textbf{(15)}$$

It is known that the Mathieu equation can have a periodic solution if the parameters {q, a} of the problem lie in certain restricted regions of the parameter space . The boundaries of the stable regions are determined by the characteristic values $\{a_n, b_n\}$ of the Mathieu equation. Fig. 1 shows a plot of these characteristic values as a function of q.

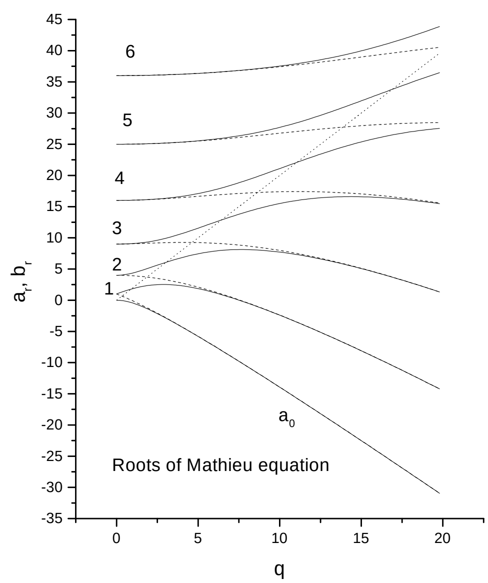

Fig.1 Characteristic values of the Mathieu equation. The solid (dashed) line shows the characteristic value a_r (b_r). The dotted line gives the constraint between the parameters a and q for this problem.

Fig. 2 shows a portion of the parameter space in greater detail. Periodic (i.e. stable) solutions exist in regions of the plot where [6]

$$a_n < a(q) < b_{n+1} \tag{16}$$

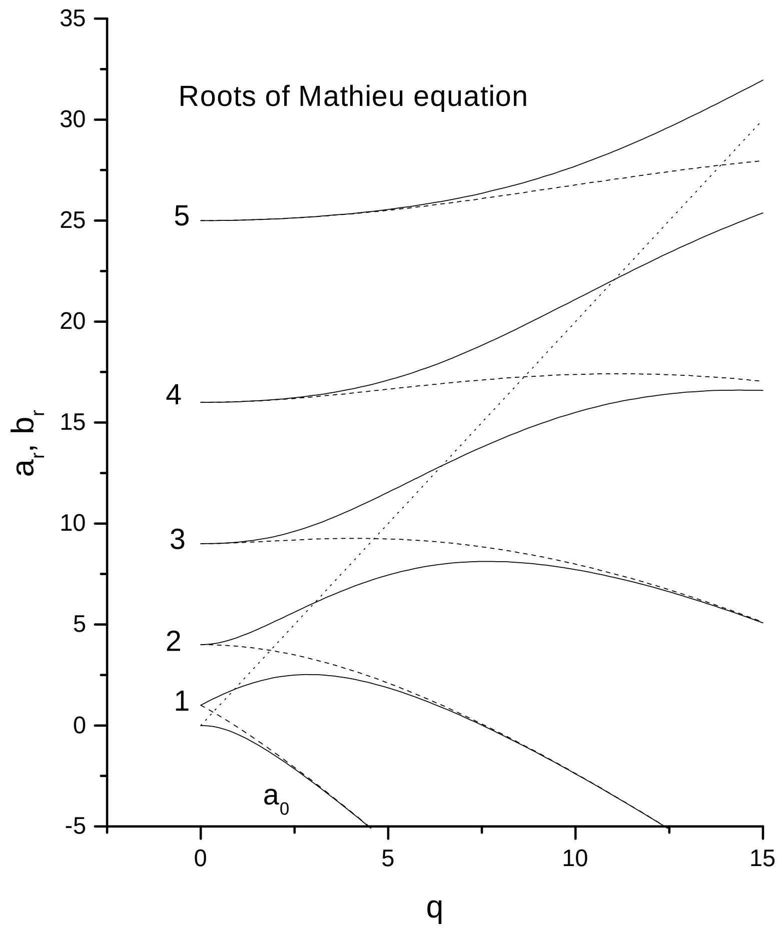

Fig.2 Characteristic values of the Mathieu equation in greater detail. The solid (dashed) line shows the characteristic value a_r (b_r). The dotted line gives the constraint between the parameters a and q for this problem.

We have tested this prediction by tracking particles through a field given by

$$\begin{aligned} B_z(r,z) &= B_o \sin(kz - a)\ I_o(kr) \\ B_r(r,z) &= B_o \cos(kz - a)\ I_1(kr) \end{aligned} \qquad \textbf{(17)}$$

where I_0 and I_1 are modified Bessel functions. This field pattern satisfies Maxwell's equations. The particle momentum was fixed at 200 MeV/c, the period of the magnetic field was 2 m and the total length of lattice was 30 m. The initial particle position (transverse momentum) was selected from a gaussian distribution with a standard deviation of 1 cm (10 MeV/c). A particle was considered to be lost if the radius ever exceeded 15 cm. The magnetic field strength B_o was chosen to give a parameter q values in the center of alternating stable and unstable bands. Transmissions were determined using 500 tracks. Results using the Monte Carlo tracking code ICOOL are shown in Table 1.

Table 1 Transmission through sinusoidal lattice

q	theory	B_o [T]	Tr [%]
0.179	S(table)	3.544	100
0.536	U(nstable)	6.133	0
1.429	S	10.02	100
2.321	U	12.76	0
3.93	S	16.61	100
5.36	U	19.40	0
7.50	S	22.94	100
9.46	U	25.77	0
12.23	S	29.30	100
14.46	U	31.86	0

We see there is perfect agreement with the predicted stability for the pure sinusoidal lattice.

3 Wavelength of particle's betatron motion

From Eq. 6 we see that the radial motion is given by the equation

$$\frac{d^2 r}{dz^2} - r \left(\frac{e B_o}{2p} \right)^2 \sin^2 kz = 0 \tag{18}$$

In general the particles undergo complicated trajectories through the lattice. Let us assume however that the motion can be broken into a slow betatron motion combined with a fast motion characterized by the lattice periodicity [8].

$$r(z) = r_S(z) - \xi_1 \sin kz - \xi_2 \cos kz \tag{19}$$

This should be a reasonable assumption so long as the betatron wavelength is large compared with the lattice period. Substituting Eq. 19 into Eq. 18, we find

$$r_S'' - \xi_1 k^2 S - \xi_2 k^2 C - \frac{k_o^2 S^2}{4} (r_S - \xi_1 S - \xi_2 C) = 0 \tag{20}$$

where

$$\begin{aligned} k_o &= \frac{e B_o}{p} \\ S &= \sin kz \\ C &= \cos kz \end{aligned} \tag{21}$$

Now assume z is a linear function of time. If we take the time average of Eq. 20, the term proportional to S^2 gives a factor of ½, while the other terms multiplied by S or C average to 0. Then Eq. 20 reduces to

$$r_S'' - \frac{k_o^2}{8} r_S = 0 \tag{22}$$

Since by definition the solution of the slow equation defines the betatron motion, we relate

$$k_\beta^2 = \left(\frac{2\pi}{\lambda} \right)^2 = \frac{k_o^2}{8} \tag{23}$$

The spatial period Λ of the betatron motion is then given by

$$\Lambda = \frac{2\pi\sqrt{8}}{e}\,\frac{p}{B_o} = 59.2\,\frac{p}{B_o} \tag{24}$$

where in the second equation the units are {m, GeV/c, T}.

4 Stability of the baseline alternating solenoid field

The actual B_Z magnetic field pattern for the 15 T, 3 m period, alternating solenoid transverse cooling section is shown in Fig. 3 as a function of z.

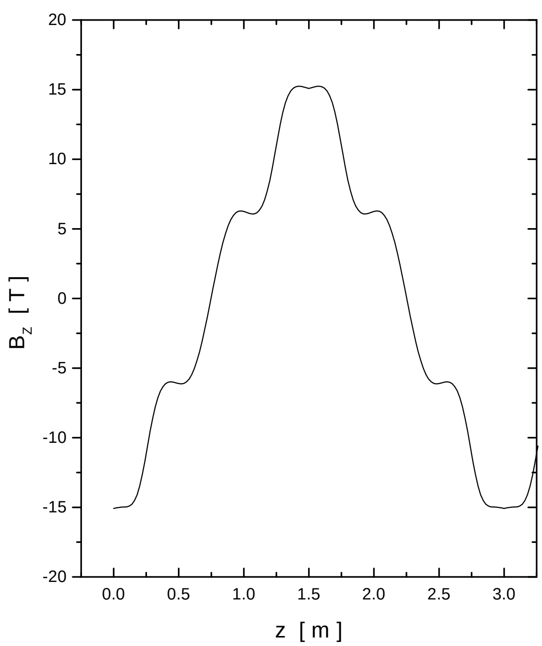

Fig. 3 Longitudinal field in the muon collider alternating solenoid solution as a function of z.

Significant deviations from a pure sine wave were required in order to match the beam from one solenoid to the next and to obtain reasonable cooling factors in the complete system. A Fourier decomposition of the field pattern is shown in Fig. 4. The dominant amplitude occurs as expected at a frequency of 1/3 m, but the pattern has a significant harmonic content.

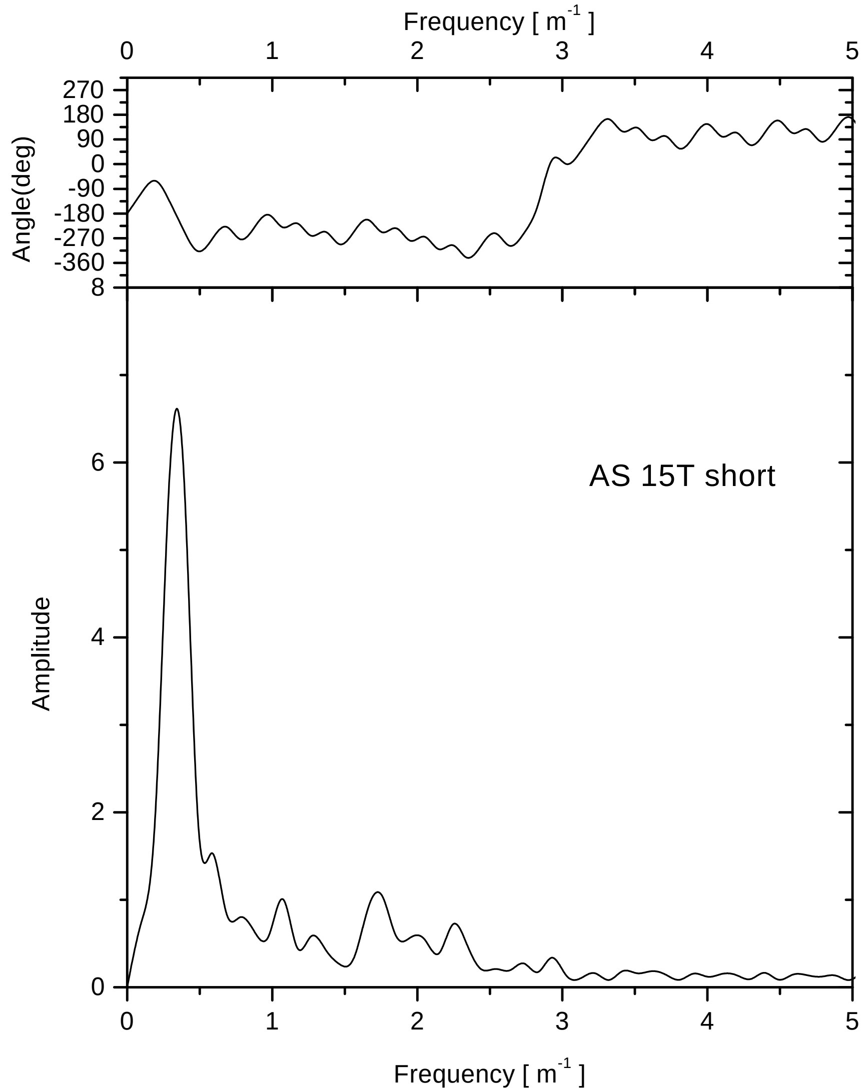

Fig. 4 Fourier decomposition of the muon collider alternating solenoid magnetic field.

We have checked the transmission down 30 m of this lattice using the code ICOOL. For convenience the peak magnetic field was fixed at 15 T and the period of the magnetic field was 3 m. The initial particle position (transverse momentum) was selected from a gaussian distribution with a standard deviation of 1 cm (15 MeV/c). A particle was considered to be lost if the radius ever exceeded 11 cm. The particle momentum was chosen to give a parameter q values in the center of alternating stable and unstable bands. Transmissions from 400 tracks are shown in Table 2.

Table 2 Transmission through AS 15T short lattice

q	theory	p [GeV/c]	Tr [%]
0.179	S(table)	1.269	100
0.536	U(nstable)	0.733	0
1.429	S	0.449	93.8
2.321	U	0.352	99.0
3.93	S	0.271	93.0
5.36	U	0.232	99.8
7.50	S	0.196	99.0
9.46	U	0.175	98.8
12.23	S	0.153	98.0
14.46	U	0.141	97.3

We see that, apart from the first predicted unstable value at q = 0.536, there is no longer any correlation of the transmission through the lattice with the predicted stable and unstable regions. There are regions where the transmission has minor dips to ~93%. Apparently the harmonic content of the magnetic field is strong enough to smear out the precise relationships among the variables required to maintain the unstable resonances. In a muon collider there will be additional sources of smearing from stochastic processes in the cooling materials and from interactions with the *rf* system. This is good news for the design of a cooling stage, since it means that, other than avoiding the first strong resonance, we can optimize the parameters $\{p, B_o, \lambda\}$ entirely in terms of the cooling performance.

5 Comparison with resonance conditions

We can rewrite Eq. 15 as

$$\frac{\lambda B_o}{p} = \frac{8\pi}{e}\sqrt{q} = 83.8\ \sqrt{q} \quad \textbf{(25)}$$

where the units are {T, m, GeV/c }. The central stable and unstable values of this quantity are

stable = { 35.5, 100.2, 166.1, 229.5, 293.1, 363.3, ... }

unstable = { 61.3, 127.7, 194.0, 257.7, 318.7, ... }

Intuitively, we expect resonant loss of particles when a particle orbit takes the same path through each magnet in the lattice. This occurs when an integer number of particle betatron wavelengths fit exactly in a period of the lattice, or

$$\lambda = n\,\Lambda \quad \textbf{(26)}$$

where n is an integer. Using Eq. 24 for Λ, we can write this as

$$\frac{\lambda B_o}{p} = 59.2\ n \quad \textbf{(27)}$$

Resonant loss of particles would then occur for

unstable = { 59.2, 118.4, 177.6, 236.8, 296.0, 355.2, ... }

It is interesting that this sequence gets increasingly out of phase from the one from the Mathieu equation. By the fifth value it predicts an unstable resonance at a location near the center of a stable band of the Mathieu equation. Recall that Table 1 showed perfect transmission for this band for a pure sinusoidal magnetic field. This is almost certainly caused by the failure of Eq. 24 to give an accurate value for Λ for these higher order resonances.

Notes and references

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